On asymptotic growth of solutions of C_0 semigroups

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Notation

$$\dot{x}(t) = Ax(t), \quad x(t) \in D(A) \subset X, t \geq 0, \tag{1}$$

X - Banach space,

$A : D(A) \to X$ - closed operator, generator of C_0-semigroup,

$\{T(t)\}_{t \geq 0}$ - C_0-semigroup generated by operator A,

$T(t)x,\ t \geq 0$ - solution of eq. (1),

$\rho(A)$ - resolvent set, e.i. $\lambda \in \mathbb{C} : (A - \lambda I)$ exists in $\mathcal{L}(X)$,

$\sigma(A) := \mathbb{C} \setminus \rho(A)$ - spectrum of A,

$R(A, \lambda) := (A - \lambda I)^{-1}$ - resolvent operator.

Outline

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1. Asymptotic Stability
2. Stability on dense subsets
3. Maximal asymptotics
4. Polynomial Stability

On Asymptotic Stability

Theorem 1 (Sklyar-Shirman '82, Arendt-Batty, Lyubich-Phong '88)

Let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X and let

$$\sigma(A) \cap (i\mathbb{R}) \quad \text{be at most countable.}$$

Then the semigroup $\{T(t)\}_{t\geq 0}$ is **strongly asymptotically stable** i.e.,

$$\lim_{t\to+\infty} \|T(t)x\| = 0 \quad \text{for all } x \in X$$

if and only if the adjoint operator A^* **has no pure imaginary eigenvalues**.

Let $\{T(t)\}$ be a C_0-semigroup on Banach space X, then

$$\forall_{\varepsilon>0} \quad \exists_{M>0} \quad \|T(t)\| \leq Me^{(\omega_0+\varepsilon)t}, \quad t \geq 0,$$

where $\omega_0 = \lim\limits_{t\to+\infty} t^{-1} \ln \|T(t)\|$ or equivalently

$$\omega_0 = \inf\{\omega \in \mathbb{R} : \exists_{M>0} \ \|T(t)\| \leq Me^{\omega t}, t \geq 0\}.$$

If $\omega_0 < 0$ then for some constants $\gamma, M > 0$ holds $\|T(t)\| \leq Me^{-\gamma t}$.
If $\omega_0 = 0$ then $\|T(t)\| \not\to 0$ and Theorem 1 implies under certain conditions that $\|T(t)x\| \to 0, x \in X$.

But in this case there is no function $g(t) \to 0, t \to +\infty$ such that

$$\|T(t)x\| \leq g(t) \cdot M_x, \quad t \geq 0, \ x \in X.$$

This means the solutions $T(t)x$ tend to zero arbitrary slow.

Outline

Theorem 2 (Batty, 1990)

Let $\{T(t)\}_{t\geq 0}$ be **bounded** C_0-semigroup on Banach space X with generator A. If in addition

$$\sigma(A) \cap (i\mathbb{R}) = \emptyset$$

Then

$$\|T(t)A^{-1}\| \to 0, \qquad t \to +\infty. \tag{2}$$

Remark

Condition (2) is equivalent to the existence of function $g(t) \to 0$, namely $g(t) := \|T(t)A^{-1}\|$ such that

$$\|T(t)x\| \leq g(t) \cdot \|x\|_{D(A)}, \quad t \to 0, \quad x \in D(A).$$

For a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on Banach space X with generator A, such that $\omega_0 = 0$:

$$\sigma(A) \subset \{\mathrm{Re}\,\lambda < 0\} \tag{3}$$

$$\Downarrow$$

(Thm. 1, Sk-Sh, Ar-Ba, Lu-Ph): all orbits $T(t)x$ tend to zero but there is no function $g(t) \to 0, t \to +\infty$ such that

$$\|T(t)x\| \leq g(t) \cdot M_x, \quad t \geq 0, \quad x \in X,$$

(Thm. 2, Batty): function $g(t) := \|T(t)A^{-1}\| \to 0$, what means that

$$\|T(t)x\| \leq g(t) \cdot \|x\|_{D(A)}, \quad t \geq 0, \quad x \in D(A).$$

Batty and Duyckaerts showed also in 2008 that condition (3) is necessary for $\|T(t)A^{-1}\| \to 0$ in case of bounded semigroups.

We show that the relations between location of spectrum (3) and stability on the domain of generator (2) can be derived from Th. 1. In addition our approach does not exploit the assumption of boundedness of the semigroup, which allows to prove necessity of condition (3) also for unbounded semigroups. Namely we prove

Theorem 3 (Sklyar, VJM 2015)

Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on Banach space X with generator A. Then for any $\lambda \notin \sigma(A)$

$$\text{a)} \quad \|T(t)(A-\lambda I)^{-1}\| \to 0 \quad \Longrightarrow \quad \sigma(A) \subset \{\mathrm{Re}\,\lambda < 0\}.$$

If in addition semigroup $T(t)$ is bounded then

$$\text{b)} \quad \|T(t)(A-\lambda I)^{-1}\| \to 0 \quad \Longleftarrow \quad \sigma(A) \subset \{\mathrm{Re}\,\lambda < 0\}.$$

Outline

Definition (Sklyar, 2010)

We say that equation $\dot{x} = Ax$ (or the semigroup $\{T(t),\ t \geq 0\}$) has a maximal asymptotics if there exists a real positive function $f(t), t \geq 0$, such that

(i) for any initial vector $x \in X$ the function $\frac{\|T(t)x\|}{f(t)}$ is bounded on $[0, +\infty)$,

(ii) there exists at least one $x_0 \in X$ such that

$$\lim_{t \to +\infty} \frac{\|T(t)x_0\|}{f(t)} = 1.$$

Theorem 4, on Maximal Asymptotics (Sklyar, 2010)

Assume that

i) $\sigma(A) \cap \{\lambda : \operatorname{Re}\lambda = \omega_0\}$ is at most countable;

ii) operator A^* does not possess eigenvalues with real part ω_0.

Then equation $\dot{x} = Ax$ (the semigroup $\{T(t),\ t \geq 0\}$) does not have any maximal asymptotics.

Remark

Above Theorem is a generalization of Theorem 1 on the case of unbounded semigroups.

Corollary 1

If the set $\sigma(A) \cap \{\lambda : \mathrm{Re}\lambda = \omega_0\}$ is empty then equation $\dot{x} = Ax$ does not have any maximal asymptotics.

Corollary 2

Let the assumptions of Theorem 4 be satisfied (lack of Max. Asympt.) and let $f(t)$, $t \geq 0$ be a positive function such that:

a) $\log f(t)$ is concave,

b) for any $x \in X$ the function $\|T(t)x\|/f(t)$ is bounded.

Then

$$\lim_{t \to +\infty} \|T(t)x\|/f(t) = 0, \quad x \in X.$$

Theorem 5 (Polak-Sklyar 2018)

Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of operators acting on Banach space X, with generator A and $\omega_0(T)=0$. Assume

(A) for any $\lambda\in\sigma(A)\cap(i\mathbb{R})$ there exists a closed and bounded component of $\sigma(A)$, say σ_λ, containing λ (i.e. $\lambda\in\sigma_\lambda\subset\sigma(A)$) and regular bounded curve Γ_λ enclosing σ_λ, such that $\Gamma_\lambda\cap\sigma(A)=\emptyset$.

(B) for any $\lambda\in\sigma(A)\cap(i\mathbb{R})$ and $x\in X_\lambda$

$$\lim_{t\to+\infty}\frac{\|T(t)x\|}{\|T(t)\|}\to 0,$$

where X_λ is an image of the Riesz projection corresponding to the curve Γ_λ.

Then the semigroup $T(t)$ has no maximal asymptotics.

Outline

1. Asymptotic Stability
2. Stability on dense subsets
3. Maximal asymptotics
4. Polynomial Stability

Consider abstract C_0-semigroup $\{T(t)\}_{t\geq 0}$.
Assume the solutions are uniformly asymptotically stable, i.e. there exists a certain function $g(t) \to 0$ such that

$$\|T(t)x\| \leq g(t) \cdot M_x, \quad t \geq 0, \quad x \in D(A).$$

Question: What is the rate of decay of the function $g(t)$?

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Question: What is the rate of decay of the function $g(t)$?

Definition (Bátkai, Engel, Prüss, Schnaubelt, 2006)

We call the semigroup (or corresponding equation) **polynomially stable**, if there exist constants $M, \alpha, \beta > 0$, such that

$$\|T(t)x\| \leq Mt^{-\beta}\|x\|_{D(A^\alpha)}, \quad t \geq 0, x \in D(A^\alpha),$$

or equivalently

$$\|T(t)A^{-\alpha}\| \leq Mt^{-\beta}, \quad t \geq 0.$$

Theorem 6 (Borichev, Tomilov, 2010)

Let $\{T(t)\}_{t\geq 0}$ be a **bounded** C_0-semigroup on Hilbert space H with generator A. If $i\mathbb{R} \subset \rho(A)$ then for any $\alpha > 0$

$$\|T(t)A^{-1}\| = O\left(t^{-\frac{1}{\alpha}}\right), \quad t \to +\infty,$$

$$\Updownarrow$$

$$\|T(t)A^{-\alpha}\| = O(t^{-1}), \quad t \to +\infty,$$

$$\Updownarrow$$

$$\|R(A, is)\| = O(|s|^{\alpha}), \quad s \to \pm\infty. \tag{4}$$

Theorem 7 (Bátkai, Engel, Prüss, Schnaubelt, 2006)

Let A generate bounded C_0-semigroup on Banach space X, and let $\sigma(A) \subset \mathbb{C}_-$. If

$$\|R(A, is)A^{-\alpha}\| \leq C, \quad s \in \mathbb{R}, \tag{5}$$

then there exists $\delta > 0$ such that

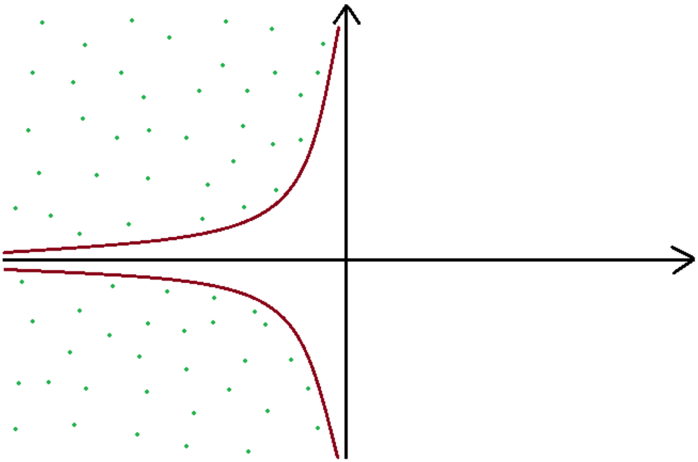
$$|\mathrm{Im}\lambda| \geq C|\mathrm{Re}\lambda|^{-\frac{1}{\alpha}}, \tag{6}$$

for $\lambda \in \sigma(A) : |\mathrm{Re}\lambda| < \delta$.

Remark

In the work (Latushkin-Shvydkoy, 2000) it is shown that the conditions (5) and (4) on the behavior of the resolvent are equivalent. Moreover, in the case of a bounded semigroup on Hilbert space the condition (6) is necessary for polynomial stability, as the consequence of Theorems 6 and 7.

Condition $|\mathrm{Im}\lambda| \geq C|\mathrm{Re}\lambda|^{-\frac{1}{\alpha}}$



Unbounded semigroups

Assumptions

(A1) $A : D(A) \subset H \to H$, generates C_0 group in Hilbert space H.

(A2) $\sigma^{(p)}(A) = \bigcup_{k\in\mathbb{Z}} \sigma_k$, such that
(a) $\sigma_i \cap \sigma_j = \emptyset$ dla $i \neq j$,
(b) $\#\sigma_k \leq N, \quad k \in \mathbb{Z}$,
(c) $\inf\{|\lambda - \mu| : \lambda \in \sigma_i, \mu \in \sigma_j, i \neq j\} = d > 0$,

(A3) linear span of generalized eigenvectors of operator A is dense in H.

Theorem 8 (Zwart, 2010)

Let generator $A : D(A) \to H$ satisfies assumptions (A1)-(A3). The family of subspaces $\{P_k(H)\}_{k\in\mathbb{Z}}$ (P_k - Riesz projection corresponding to σ_k) forms a Riesz basis of subspaces, i.e. there exists constants $m, M > 0$, such that for any $x \in H$ holds

$$m\|x\|^2 \leq \sum_{k\in\mathbb{Z}} \|P_k x\|^2 \leq M\|x\|^2.$$

Theorem 9 (Sklyar-Polak, 2016)

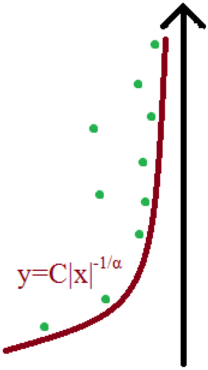
Let A satisfies assumptions (A1)-(A3), $\sigma(A) \subset \mathbb{C}_-$ and for some constants $C, \alpha > 0$ holds

$$|\mathrm{Im}\,\lambda| \geq \mathrm{C}|\mathrm{Re}\lambda|^{-\frac{1}{\alpha}} : \lambda \in \sigma(\mathcal{A}).$$

Then

$$\|T(t)A^{-N\alpha}\| = O(\frac{1}{t}), \quad t > 0,$$

$$\|R(A, is)\| = O(|s|^{N\alpha}), \quad s \to \pm\infty.$$



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Thank You for Your attention